THE QUESTIONS OF MOTION FOR AN EARTH SATELLITE TAKING INTO ACCOUNT ATMOSPHERIC DRAG

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The equations of motion for an Earth satellite were derived in the paper [1]. These equations were used to investigate the effects of the noncentral character of the Earth gravitational field. The present derivation of the equations of motion for a satellite takes into account atmospheric drag. As was indicated in [2] (p.92), the atmospheric effects are difficult to take into account for many methods of trajectory computation.

1. The notation adopted in [1] is used in the present paper. The origin of a fixed system of coordinates O, x, y, z with unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ is located at the center of the Earth. The z-axis is directed toward the north pole. The location of a point $N(x, y, z)$ is determined by means of the spherical coordinates r, ϑ, λ

$$x = r\sin\vartheta\cos\lambda, \qquad y = r\sin\vartheta\sin\lambda, \qquad z = r\cos\vartheta \tag{1.1}$$

The coordinate trihedron of the spherical system of coordinates is denoted by $\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\lambda$. A point with unit mass (satellite) is moving under the action of the forces of attraction and atmospheric resistance. The potential energy of the point N is of the form ([2], p.75)

$$\Pi(u, \gamma) = -\mu u - \varepsilon u^3(1 - 3\gamma^2) - \ldots \qquad (u = r^{-1},\ \gamma = \cos\vartheta) \tag{1.2}$$

It is assumed that the Earth's atmosphere rotates about the z-axis with angular velocity $\Omega(u, \gamma)$ which depends on $u = r^{-1}$ and $\gamma = \cos\vartheta$.

It is regarded that the force F of atmospheric resistance is directed opposite to the satellite velocity relative to the atmosphere

$$\mathbf{F} = -\rho(u, \gamma, v)(\mathbf{v} - \mathbf{\Omega}\times\mathbf{r}), \qquad \mathbf{\Omega} = \mathbf{i}_3\Omega \qquad \left(\mathbf{v} = \frac{d\mathbf{r}}{dt}\right) \tag{1.3}$$

Here ρ is the experimental proportionality coefficient, $\mathbf{r}$ is the radius vector of the point N. The derivation utilizes the orbital set $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{n}$, where

$$\mathbf{e}_r = \mathbf{r}r^{-1}, \qquad \mathbf{n} = (\mathbf{r}\times d\mathbf{r}/dt)\,|\mathbf{r}\times d\mathbf{r}/dt|^{-1}, \qquad \mathbf{e}_\varphi = \mathbf{n}\times\mathbf{e}_r \tag{1.4}$$

2. The satellite equation of motion

$$d^2r/dt^2 = -\operatorname{grad}\Pi - \rho(dr/dt - \mathbf{\Omega}\times\mathbf{r}) \tag{2.1}$$

is transformed into a form convenient for computation. Let us introduce the angular momentum vector

$$\mathbf{k} = \mathbf{r}\times d\mathbf{r}/dt, \qquad \mathbf{k} = \mathbf{n}k \tag{2.2}$$

Substituting the vector $\mathbf{r} = r\mathbf{e}_\varphi$ into (2.1) and dot multiplying by $\mathbf{e}_r$ we get

$$d^2r / dt^2 - k^2 / r^2 = -\partial\Pi / \partial r - \rho dr / dt \tag{2.3}$$

Introduce a change of variables

$$u = r^{-1}, \qquad d\varphi = u^2 dt = r^{-2} dt \tag{2.4}$$

In view of (2.4), Equation (2.3) is transformed into the form

$$d^2u / d\varphi^2 + (\rho / u^2) du / d\varphi + hu = -\partial\Pi / \partial u \qquad (h = k^2) \tag{2.5}$$

Substituting $\mathbf{r} = r\mathbf{e}_r$ into (2.2) and cross multiplying by $\mathbf{e}_r$ we get

$$d\mathbf{e}_r / dt = r^{-2} k \times \mathbf{e}_r, \qquad d\mathbf{e}_r / d\varphi = \mathbf{k} \times \mathbf{e}_r \tag{2.6}$$

Differentiating (2.6) with respect to φ we find

$$d^2\mathbf{e}_r / d\varphi^2 + k^2 \mathbf{e}_r = d\mathbf{k} / d\varphi \times \mathbf{e}_r \tag{2.7}$$

From (2.2) and (2.1) we find

$$\begin{aligned} d\mathbf{k} / dt &= -\mathbf{r} \times ((\partial\Pi / \partial r)\, \mathbf{e}_r + 1/r\, (\partial\Pi / \partial\vartheta)\, \mathbf{e}_\vartheta) - \rho\mathbf{r} \times d\mathbf{r} / dt + \\ &+ \rho\mathbf{r} \times (\Omega \times \mathbf{r}) = (-\partial\Pi / \partial\vartheta)\, \mathbf{e}_\lambda - \rho\mathbf{k} + \rho r^2 \Omega\, [\mathbf{i}_3 - \mathbf{e}_r (\mathbf{i}_3 \mathbf{e}_r)] \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7) and in view of (2.4), we get

$$\frac{d^2\mathbf{e}_r}{d\varphi^2} + k^2 \mathbf{e}_r = -\frac{1}{u^2} \frac{\partial\Pi}{\partial\vartheta} \mathbf{e}_\vartheta - \frac{\rho}{u^2} \frac{d\mathbf{e}_r}{d\varphi} + \frac{\rho\Omega}{u^4} \mathbf{i}_3 \times \mathbf{e}_r \tag{2.9}$$

Dot multiplying (2.9) by $\mathbf{i}_3$ we obtain the equation for γ

$$\frac{d^2\gamma}{d\varphi^2} + \frac{\rho}{u^2} \frac{d\gamma}{d\varphi} + h\gamma = (\gamma^2 - 1) \frac{1}{u^2} \frac{\partial\Pi}{\partial\gamma} \qquad (h = k^2,\ \gamma = \cos\vartheta) \tag{2.10}$$

The quantity h is contained in Equations (2.5) and (2.10). We find the differential equation for h by dot multiplying (2.8) by $2\mathbf{k}$ and taking into account the substitution (2.4)

$$\frac{dh}{d\varphi} = -\frac{2}{u^2} \frac{\partial\Pi}{\partial\gamma} \frac{d\gamma}{d\varphi} - \frac{2\rho h}{u^2} + \frac{2\rho\Omega}{u^4} \left[h(1 - \gamma^2) - \left(\frac{d\gamma}{d\varphi}\right)^2 \right]^{1/2} \tag{2.11}$$

At the same tim we utilize the equalities (2.12)

$$\frac{\partial\Pi}{\partial\vartheta} \mathbf{e}_\lambda \cdot \mathbf{k} = \frac{\partial\Pi}{\partial\gamma} \frac{d\gamma}{d\vartheta} (\mathbf{i}_3 \times \mathbf{e}_r) \cdot \mathbf{k} \operatorname{cosec} \vartheta = \frac{\partial\Pi}{\partial\gamma} \mathbf{i}_3 \times (\mathbf{k} \times \mathbf{e}_r) = \frac{\partial\Pi}{\partial\gamma} \mathbf{i}_3 \cdot \frac{d\mathbf{e}_r}{d\varphi} = \frac{\partial\Pi}{\partial\gamma} \frac{d\gamma}{d\varphi}$$

$$\begin{aligned} (\mathbf{i}_3 \cdot \mathbf{k})^2 = \mathbf{k}^2 - (\mathbf{i}_3 \times \mathbf{k})^2 &= h - \left| \mathbf{i}_3 \times \left(\mathbf{e}_r \times \frac{d\mathbf{e}_r}{d\varphi} \right) \right|^2 = \\ &= h - \left| \mathbf{e}_r \left(\mathbf{i}_3 \cdot \frac{d\mathbf{e}_r}{d\varphi} \right) \right|^2 - \left| \frac{d\mathbf{e}_r}{d\varphi} (\mathbf{i}_3 \cdot \mathbf{e}_r) \right|^2 = h(1 - \gamma^2) - \left(\frac{d\gamma}{d\varphi} \right)^2 \end{aligned} \tag{2.13}$$

The system of equations (2.5), (2.10), (2.11) is of fifth order and is complete. For $\rho = 0$ the second form of the equations of motion is obtained [1]. The time t is found from (2.4) by quadrature.

3. Equation (2.9) can be utilized to find the longitude angle λ . Projecting it in the x- and y-axes we obtain the differential equations

$$\begin{aligned} \frac{d^2\gamma_1}{d\varphi^2} + \frac{\rho}{u^2} \frac{d\gamma_1}{d\varphi} + h\gamma_1 &= \frac{1}{u^2} \frac{\partial\Pi}{\partial\gamma} \gamma\gamma_1 - \frac{\Omega\rho}{u^4} \gamma_2 \\ \frac{d^2\gamma_2}{d\varphi^2} + \frac{\rho}{u^2} \frac{d\gamma_2}{d\varphi} + h\gamma_2 &= \frac{1}{u^2} \frac{\partial\Pi}{\partial\gamma} \gamma\gamma_2 + \frac{\Omega\rho}{u^4} \gamma_1 \end{aligned} \tag{3.1}$$

for the direction cosines γ_1, γ_2 where

$$\gamma_1 = \sin\vartheta \cos\lambda, \qquad \gamma_2 = \sin\vartheta \sin\lambda,\ \mathbf{e}_r = \gamma_1 \mathbf{i}_1 + \gamma_2 \mathbf{i}_2 + \gamma \mathbf{i}_3 \tag{3.2}$$

It is more convenient to determine λ from the relationships

$$\tan\lambda = \frac{\gamma_2}{\gamma_1}, \qquad \frac{d\lambda}{d\varphi} = \left(\frac{d\gamma_2}{d\varphi}\gamma_1 - \frac{d\gamma_1}{d\varphi}\gamma_2\right)(\gamma_1^2+\gamma_2^2)^{-1} =$$

$$= \left(\mathbf{e}_r \times \frac{d\mathbf{e}_r}{d\varphi}\right)\mathbf{i}_3(1-\gamma^2)^{-1} = \mathbf{k}\mathbf{i}_3(1-\gamma^2)^{-1} = \left[h(1-\gamma^2) - \left(\frac{d\gamma}{d\varphi}\right)^2\right]^{1/2}(1-\gamma^2)^{-1} \quad (3.3)$$

If γ and h are known functions of φ, then λ is found by quadrature. The angular velocity $d\lambda/d\varphi$, on the strength of (3.1), satisfies Equation

$$\frac{d}{d\varphi}\left(\frac{d\lambda}{d\varphi}\right) = \left(\frac{2\gamma}{1-\gamma^2}\frac{d\gamma}{d\varphi} - \frac{\rho}{u^2}\right)\frac{d\lambda}{d\varphi} + \frac{\Omega\rho}{u^4} \quad (3.4)$$

which can be used for checking during computations. The projection of $\mathbf{k}$ on the z-axis is

$$\varkappa = \mathbf{k}\cdot\mathbf{i}_3 = \left[h(1-\gamma^2) - \left(\frac{d\gamma}{d\varphi}\right)^2\right]^{1/2} = (1-\gamma^2)\frac{d\lambda}{d\varphi} \quad (3.5)$$

The equation for $\varkappa$ is found from (3.4) as

$$\frac{d\varkappa}{d\varphi} = -\frac{\rho}{u^2}\varkappa + \frac{\Omega\rho}{u^4}(1-\gamma^2) \quad (3.6)$$

4. As was indicated in [3] (p.209), there is a high velocity air current moving from west to east at high altitudes around the Earth. This is evidenced by the fact that the angular velocity Ω of the atmosphere at high altitudes can differ substantially from the Earth's rotation rate.

Let us find the expression for Ω by means of motion characteristics which can be determined by observation. If $\rho = 0$, then Equations (2.5) and (2.11) posses the energy integral

$$E = \Pi + {}^1/_2(du/d\varphi)^2 + {}^1/_2 hu^2 \quad (4.1)$$

For $\rho \neq 0$ the quantity E will be variable. From (2.5), (2.11) and (3.5) we find the equation for E as

$$\frac{dE}{d\varphi} = -\frac{2\rho}{u^2}(E-\Pi) + \frac{\rho\Omega}{u^2}\varkappa \quad (4.2)$$

From (3.6) and (4.2) we find Ω

$$\Omega = \left(2E - 2\Pi - \varkappa\frac{dE}{d\varkappa}\right)\left(\varkappa + \frac{1-\gamma^2}{u^2}\frac{dE}{d\varkappa}\right)^{-1} \quad (4.3)$$

The quantities E, $\varkappa$ can be found from observation of the satellite motion. These quantities change by small amounts during the motion

$$\varkappa = \mathbf{i}_3\cdot(\mathbf{r}\times\mathbf{v}), \qquad E = \Pi + {}^1/_2\mathbf{v}^2, \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} \quad (4.4)$$

Formula (4.3) is essentially simplified for equatorial, near circular trajectories when $\gamma \equiv 0$.

5. Let us consider a unit sphere with center at the origin of the coordinates O, x, y, z.

During motion the radius vector $\mathbf{r}$ of the point M generates a line on the sphere the length τ of which will be taken as the independent variable

$$d\tau = r^{-2}|\mathbf{r}\times d\mathbf{r}| = r^{-2}|\mathbf{r}\times d\mathbf{r}/dt|\,dt = r^{-2}k\,dt = (h)^{1/2}d\varphi \quad (5.1)$$

Transforming Equations (2.5), (2.10) and (2.11) with the variable τ and denoting differentiation with respect to τ by primes, we obtain the system of differential equations

$$u'' + u = -\frac{1}{h}\frac{\partial\Pi}{\partial u} + \frac{1}{hu^2}\frac{\partial\Pi}{\partial\gamma}\gamma' u' + \frac{\rho\Omega}{hu^4}(1-\gamma^2-\gamma'^2)^{1/2}u' \quad (5.2)$$

$$\gamma'' + \gamma = -(1-\gamma^2-\gamma'^2)\frac{1}{hu^2}\frac{\partial\Pi}{\partial\gamma} - \frac{\rho\Omega}{hu^4}(1-\gamma^2-\gamma'^2)^{1/2}\gamma' \quad (5.3)$$

$$h' = -\frac{2}{u^2}\frac{\partial\Pi}{\partial\gamma}\gamma' - \frac{2\rho\sqrt{h}}{u^2} + \frac{2\rho\Omega}{u^4}(1-\gamma^2-\gamma'^2)^{1/2} \quad (5.4)$$

The first form of the equations of motion [1] is obtained when $\rho = 0$.

Let us consider the plane of the orbit which contains the vectors $\mathbf{r}$ and $d\mathbf{r}/dt$. The inclination angle of the orbital plane relative to the equatorial plane will be denoted by i . From (2.13) we have

$$\cos i = \mathrm{n}\cdot\mathbf{i}_3 = \mathbf{k}\cdot\mathbf{i}_3 h^{-1/2} = (1-\gamma^2-\gamma'^2)^{1/2} \tag{5.5}$$

Denoting $l = \cos i$, we obtain from (5.3) Equation for l

$$l' = \frac{\gamma'}{hu^2}\frac{\partial\Pi}{\partial\gamma}l + \frac{\rho\Omega}{hu^4}\gamma'^2 \tag{5.6}$$

Equations (5.2) to (5.4) are especially convenient for investigation of polar orbit of the satellite when $l = \cos i \approx 0$. Similarly, we can derive the equations which correspond to the third form of the satellite equations of motion [1]. These equations are convenient for investigating the near circular trajectories.

6. As an example of use of Equations (5.2) to (5.4) we consider the question of stability of a circular equatorial orbit. Let during motion $\gamma \equiv 0$. Equations (5.2) to (5.4) become

$$u'' + gfh^{-1}u' + u = h^{-1}[\mu + 3\varepsilon u^2 + O(\varepsilon^2)], \qquad h' = 2g\,[-\sqrt{h} + f] \tag{6.1}$$

$$g = \rho u^{-2}, \qquad f = \Omega u^{-2} \tag{6.2}$$

It is noted that the quantity f depends only on u . Equations (6.1) have a stationary solution u_0, h_0 which is defined by Equations

$$u = h^{-1}[\mu + 3\varepsilon u^2 + O(\varepsilon^2)], \qquad h = f^2 \tag{6.3}$$

The value of the functions for the generating solution u_0, h_0 will be denoted by the subscript $_0$. Equations in variations for (6.1) are of the form

$$\delta u'' + g_0 f_0 h_0^{-1}\delta u' + (1 + O(\varepsilon))\,\delta u = -(\mu h_0^{-2} + O(\varepsilon))\,\delta h$$

$$\delta h' = -g_0 h_0^{-1/2}\delta h + 2g_0\frac{df_0}{du_0}\delta u \tag{6.4}$$

The characteristic equation for the system (6.4)

$$[p^2 + g_0 f_0 h_0^{-1}p + 1 + O(\varepsilon)]\,(p + g_0 h_0^{-1/2}) = -2g_0\frac{df_0}{du_0}(\mu h_0^{-2} + O(\varepsilon)) \tag{6.5}$$

has the roots p_1, p_2, p_3 which are given by

$$p_1 = -g_0 h_0^{-1/2} - 2g_0 h_0^{-2}\frac{df_0}{du_0}\mu + O(g^2+\varepsilon) \tag{6.6}$$

$$\operatorname{Re} p_{2,3} = -0.5g_0 h_0^{-1/2} + g_0 h_0^{-2}\frac{df_0}{du}\mu + O(g^2+\varepsilon)$$

The stability criterion for the circular orbit is found from (6.6) as

$$\left|\frac{df_0}{du_0}u_0 f_0^{-1}\right| < 0.5 + O(\varepsilon g_0^{-1} + g_0) \tag{6.7}$$

With (6.2) and (2.4) taken into account, the inequality (6.7) becomes

$$\left|\frac{d\ln\Omega_0}{d\ln r_0} + 2\right| < 0.5 + O(\varepsilon g_0^{-1} + g_0) \tag{6.8}$$

Let the atmosphere rotate about the Earth with a constant angular velocity Ω . The stationary solution of the system (6.1) corresponds to the circular orbit of the satellite which is at rest relative to the atmosphere. This motion will be unstable due to the action of the atmosohere since (6.8) is not fulfilled.

During the derivation of (6.8) it was assumed that the small quantity ε is a higher order than g_0 . Also, the effect of the noncentral character of the gravitational field is neglected in comparison with the effect of the atmosphere.

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ON THE PERMANENT AXES OF ROTATION OF A GYROSTAT WITH A FIXED POINT

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Permanant rotations of a heavy rigid body were discovered by Mlodzeevskii [1] and Staude [2].

The necessary conditions for the stability of permanent rotations of a heavy rigid body were investigated by Grammel [3]. The sufficient conditions for stability of permanent rotations both for a general case with arbitrary mass distribution inside the body, and for a number of special cases were derived by Rumiantsev [4]. A detailed investigation of permanent rotations of a gyrostat moving by inertia, and of its stability is due to Volterra[5]. A geometrical interpretation of the motion of a gyrostat in the latter case was given for the first time by Zhukovskii [6]. The problem of distribution of permanent axes of rotation of a heavy gyrostat has been partially solved by Anchev [7] and Drofa [8]. The necessary and sufficient conditions of stability for certain motions of heavy gyrostats were found by Rumiantsev [9].

In this work we determine the permanent axes of rotation of a gyrostat under the action of forces resulting from a force function U, and depending only on the position of the gyrostat.

We assume that the gyrostat S consists of the rigid body S_1, having a fixed point O and of the bodies S_2 joined nonpermanently with S_1. The angular momentum of the bodies S_2 in their motion with respect to the body S_1 is assumed to be constant. We shall investigate the stability of certain motions of the gyrostat using the second method of Liapunov.

1. The orientation of the rectangular axes $OXYZ$ determine the position of the gyrostat S with the fixed point O. The axes $OXYZ$ are fixed in